ITERATIVE APPROXIMATIONS OF EXPONENTIAL BASES ON FRACTAL MEASURES

DORIN ERVIN DUTKAY, DEGUANG HAN, AND ERIC WEBER

Abstract. For some fractal measures it is a very difficult problem in general to prove the existence of spectrum (respectively, frame, Riesz and Bessel spectrum). In fact there are examples of extremely sparse sets that are not even Bessel spectra. In this paper we investigate this problem for general fractal measures induced by iterated function systems (IFS). We prove some existence results of spectra associated with Hadamard pairs. We also obtain some characterizations of Bessel spectrum in terms of finite matrices for affine IFS measures, and one sufficient condition of frame spectrum in the case that the affine IFS has no overlap.

Contents

1. Introduction

Joseph Fourier introduced Fourier series on the interval $[0,1)$ (or $[-\pi,\pi)$), i.e. expansions of functions as a series with terms $e^{2\pi inx}$. These exponentials are now understood as an orthonormal basis for the Hilbert space $L^2[0,1]$ with respect to Lebesgue measure. If one considers a Borel probability measure μ on $[0,1]$ (or $\mathbb{R}^d$ more generally) other than Lebesgue measure, a natural question is whether there exists an orthonormal basis for $L^2(\mu)$ of the form $e^{2\pi i\lambda_n x}$, for some sequence of frequencies $\{\lambda_n\} \subset \mathbb{R}$. If μ does have such a Fourier basis, it is called a *spectral measure*. One of the first examples of a singular measure which is spectral was given by Jorgensen and Pedersen in [JP98] which was a measure on a Cantor like set. Also in [JP98] is a proof that on the usual Cantor middle third set, the natural measure does not have a (orthogonal) Fourier basis.

Since this measure does not have an orthonormal basis of exponentials, we then consider whether this measure has a non-orthogonal basis of exponentials, such as a Riesz basis [PW87], Schauder basis [Sin70] or even a frame of exponentials introduced by Duffin and Schaeffer [DS52] in the context of nonharmonic Fourier series. For the Hilbert space $L^2[0,1]$ with respect to Lebesgue

2000 *Mathematics Subject Classification.* 28A80,28A78, 42B05.

Key words and phrases. fractal, iterated function system, frame, Bessel sequence, Riesz basic sequence, Beurling dimension.

This work is partially supported by the NSF grant DMS-1106934.

measure, the main result of Duffin and Schaeffer is a sufficient density condition for $\{e^{2\pi i\lambda x}\}_{\Lambda\in\Lambda}$ to be a frame. An almost complete characterization of the frame properties of $\{e^{2\pi i\lambda x}\}_{\lambda\in\Lambda}$ in terms of lower Beurling density was obtained by Jaffard, [Jaf91] and Seip [Sei95], with a complete characterization given in [OCS02] in terms of the zero sequence of certain Hermite-Biehler functions. Moreover, in this classical case, the Bessel sequences correspond the sets Λ that have finite upper Beurling density, and $\{e^{2\pi i\lambda x}\}_{\lambda\in\Lambda}$ is a Riesz sequences if Λ is a sparse enough in the sense that its upper Beurling density is strictly less than 1. Therefore it is very easy to construct, Bessel sequences/frames/Riesz sequences of exponentials for the unit interval. However this is not the case anymore for fractal measures. For some fractal measures it is difficult even to prove the existence of frames or Riesz bases. For example, it is still an open problem whether there exists a frame or Riesz basis for the fractal measure on the Cantor middle third set. In fact for this measure there exists a set Λ which is extremely sparse but is not a Bessel spectrum [DHSW11]: For any integer $a\in\mathbb{Z}\setminus\{0\}$, and any infinite set of non-negative integers F, the set $\{3^n a\,|\,n\in F\}$ cannot be a Bessel spectrum for the middle-third Cantor measure μ_3. Moreover, any such set has upper Beurling dimension 0. For this reason it seems that it is not even clear how to construct a infinite sequence Λ such that its corresponding exponentials form a Bessel sequence or Riesz sequence. We investigate this problem in this paper for general fractal measures induced by iterated function systems (IFS).

Definition 1.1. A sequence $\{x_n\}_{n=1}^{\infty}$ in a Hilbert space (with inner product $\langle\cdot,\cdot\rangle$) is *Bessel* if there exists a positive constant B such that for all v

$$\sum_{n=1}^{\infty}|\langle v,x_n\rangle|^2\leq B\|v\|^2.$$

This is equivalent to the existence of a positive constant D such that for every finite sequence $\{c_1,\dots,c_K\}$ of complex numbers

$$\|\sum_{n=1}^{K}c_nx_n\|\leq D\sqrt{\sum_{n=1}^{K}|c_n|^2}.$$

Here $D^2=B$ is called the Bessel bound.

The sequence is a frame if in addition to being a Bessel sequence there exists a positive constant A such that for all v

$$A\|v\|^2\leq\sum_{n=1}^{\infty}|\langle v,x_n\rangle|^2\leq B\|v\|^2.$$

In this case, A and B are called the lower and upper frame bounds, respectively.

The sequence is a Riesz basic sequence if in addition to being a Bessel sequence there exists a positive constant C such that for every finite sequence $\{c_1,\dots,c_K\}$ of complex numbers

$$C\sqrt{\sum_{n=1}^{K}|c_n|^2}\leq\|\sum_{n=1}^{K}c_nx_n\|\leq D\sqrt{\sum_{n=1}^{K}|c_n|^2}.$$

Here C and D are called the lower and upper basis bounds, respectively.

A sequence $\{x_n\}$ in a Banach space is a Schauder basic sequence if there exists a constant E such that for every finite sequence $\{c_1, \ldots, c_K\}$ of complex numbers,

$$\left\| \sum_{n=1}^{k} c_n x_n \right\| \leq E \left\| \sum_{n=1}^{K} c_n x_n \right\|$$

where $1 \leq k \leq K$. E is called the basis constant.

In what follows we denote by

$$e_\lambda(x) := e^{2\pi i \lambda \cdot x}, \quad (x \in \mathbb{R}^d)$$

for $\lambda \in \mathbb{R}^d$. We will also denote by $\widehat{\mu}$, the Fourier transform of a measure μ on $\mathbb{R}^d$:

$$\widehat{\mu}(x) = \int e^{2\pi i x \cdot t}\, d\mu(t), \quad (x \in \mathbb{R}^d)$$

Definition 1.2. A Borel probability measure μ on $\mathbb{R}^d$ is called spectral if there exists a subset Λ of $\mathbb{R}^d$ such that $E(\Lambda) := \{e_\lambda : \lambda \in \Lambda\}$ is an orthonormal basis for $L^2(\mu)$. In this case, Λ is called a *spectrum* for the measure μ, and (μ, Λ) is called a *spectral pair*.

A subset Λ of $\mathbb{R}^d$ is said to be *orthogonal* in $L^2(\mu)$ if the corresponding set of exponential functions $E(\Lambda)$ is orthogonal in $L^2(\mu)$. The set Λ is called a *Bessel/frame/Riesz basic/Schauder basic spectrum* if the set of exponentials $E(\Lambda)$ is a Bessel sequence/frame/Riesz basic/Schauder basic sequence. We say that the corresponding Bessel/frame/Riesz basis bounds are the bounds for Λ.

We will sometimes identify the set Λ with the corresponding set of exponential functions $E(\Lambda)$; for example, we will say that Λ spans a subspace if $E(\Lambda)$ spans it.

Definition 1.3. Consider a $d \times d$ expansive integer matrix R and a subset $B \subset \mathbb{Z}^d$ with $\#B = N \geq 2$. We define the iterated function system by

$$\tau_b(x) = R^{-1}(x + b), \quad (x \in \mathbb{R}^d).$$

We also define the following operator $\mathcal{T}$ on Borel probability measures on $\mathbb{R}^d$

$$(\mathcal{T}\mu)(E) = \frac{1}{N} \sum_{b \in B} \mu(\tau_b^{-1}(E)), \tag{1.1}$$

for any Borel probability measure μ and all Borel sets E. Equivalently the measure $\mathcal{T}\mu$ is defined by

$$\int f\, d\mathcal{T}\mu = \frac{1}{N} \sum_{b \in B} \int f \circ \tau_b\, d\mu, \tag{1.2}$$

for all continuous functions f on $\mathbb{R}^d$.

We denote by μ_B the unique invariant measure for the operator $\mathcal{T}$, i.e., $\mathcal{T}\mu_B = \mu_B$. (See [Hut81] for the existence and uniqueness of this measure)

We say that the measure μ_B has no overlap if

$$\mu_B(\tau_b(X_B) \cap \tau_{b'}(X_B)) = 0 \text{ for all } b \neq b' \text{ in } B.$$

For a subset L of $\mathbb{R}^d$, define the operator $\mathcal{S}$ by

$$\mathcal{S}\Lambda = \bigcup_{l\in L}(R^T\Lambda + l).$$

2. Orthogonal exponentials

We present here a technique for constructing an orthogonal sequence of exponentials for an invariant measure for an IFS via a sequence of orthogonal exponentials for a sequence of approximating measures. What is required is that the IFS possesses a dual IFS in the sense of Hadamard pairs. We begin with an orthogonal set of exponentials Λ_0 for an initial measure μ_0, and the let the IFS act on μ_0 and the dual IFS act on Λ_0, then consider the limit of this process in a specific manner. For dimension 1, in the limit we obtain that the invariant measure for the IFS is a spectral measure and construct explicitly a spectrum for that measure (Theorem 2.8).

Definition 2.1. Let L be a subset of $\mathbb{R}^d$ with $\#L = \#B = N$. We say that (B, L) is a Hadamard pair if the matrix

$$\frac{1}{\sqrt{N}}\left(e^{2\pi i R^{-1}b\cdot l}\right)_{b\in B, l\in L}$$

is unitary.

Proposition 2.2. *Let (B, L) be a Hadamard pair. Define the set*

$$\Pi(B) = \left\{\gamma \in \mathbb{R}^d : \gamma\cdot b \in \mathbb{Z} \text{ for all } b \in B\right\}.$$

If μ is a spectral measure with spectrum Λ contained in $\Pi(B)$, then $\mathcal{T}\mu$ is a spectral measure with spectrum $\mathcal{S}\Lambda$.

We need some lemmas:

Lemma 2.3. [DJ06] *Let μ_B be the invariant measure for the IFS $(\tau_b)_{b\in B}$, i.e., $\mathcal{T}\mu_B = \mu_B$. Then*

$$\widehat{\mu}_B(x) = m_B((R^T)^{-1}x)\widehat{\mu}_B((R^T)^{-1}x), \quad (x \in \mathbb{R}^d) \tag{2.1}$$

where

$$m_B(x) = \frac{1}{N}\sum_{b\in B} e^{2\pi i b\cdot x}\,(x \in \mathbb{R}^d) \tag{2.2}$$

A set Λ is a spectrum for a Borel probability measure μ iff

$$\sum_{\lambda\in\Lambda}|\widehat{\mu}(x+\lambda)|^2 = 1, \quad (x \in \mathbb{R}^d) \tag{2.3}$$

Definition 2.4. For a measure μ on $\mathbb{R}^d$ and a subset Λ of $\mathbb{R}^d$, define

$$h_{\mu,\Lambda}(x) = \sum_{\lambda\in\Lambda}|\widehat{\mu}(x+\lambda)|^2, \quad (x \in \mathbb{R}^d)$$

For a pair of subsets (B, L) of $\mathbb{Z}^d$ we define the *transfer operator* on functions on $\mathbb{R}^d$:

$$(R_{B,L}f)(x) = \sum_{l\in L}|m_B((R^T)^{-1}(x+l))|^2 f((R^T)^{-1}(x+l)), \quad (x \in \mathbb{R}^d).$$

Lemma 2.5. *Let (B, L) be a Hadamard pair.*

(i) *For any measure μ, we have*

$$\widehat{(\mathcal{T}\mu)}(x) = m_B((R^T)^{-1}x)\widehat{\mu}((R^T)^{-1}x), \quad (x \in \mathbb{R}^d)$$

(ii) *For any measure μ and any subset Λ of $\Pi(B)$, we have*

$$R_{B,L}h_{\mu,\Lambda} = h_{\mathcal{T}\mu,\mathcal{S}\Lambda}.$$

Proof. (i) follows from (1.2) by taking the Fourier transform in the definition of $\mathcal{T}\mu$.

For (ii) we compute $h_{\mathcal{T}\mu,\mathcal{S}\Lambda}$. Since (B,L) form a Hadamard pair it is easy to see that the elements of L are incongruent $\mod R^T\Pi(B)$. Therefore, if $\Lambda \subset \Pi(B)$, then the sets $R^T\Lambda + l$, $l \in L$ are disjoint. We then have, using (i):

$$h_{\mathcal{T}\mu,S\Lambda}(x) = \sum_{l\in L}\sum_{\lambda\in\Lambda}|\widehat{\mathcal{T}\mu}(x+R^T\lambda+l)|^2 = \sum_{l\in L}\sum_{\lambda\in\Lambda}|m_B\left((R^T)^{-1}(x+l)+\lambda\right)|^2\left|\widehat{\mu}\left((R^T)^{-1}(x+l)+\lambda\right)\right|^2$$
$$= R_{B,L}h_{\mu,\Lambda}(x),$$

where we used the fact that $m_B(y+\lambda) = m_B(y)$ for $\lambda \in \Pi(B)$.

□

Proof of Proposition 2.2. From Lemma 2.3 we know that we have to check that $h_{\mathcal{T}\mu,S\Lambda} = 1$, and we know that $h_{\mu,\Lambda} = 1$. But from Lemma 2.5 we have

$$h_{\mathcal{T}\mu,S\Lambda} = R_{B,L}h_{\mu,\Lambda} = R_{B,L}1 = 1.$$

In the last equality we used the fact that (B,L) is a Hadamard pair, which implies $R_{B,L}1 = 1$ (see [DJ06]). □

Corollary 2.6. *Let (B,L) be a Hadamard pair. Define the operator $\mathcal{M}$ on subsets K of $\mathbb{R}^d$ by*

$$\mathcal{M}K = \bigcup_{b\in B}\tau_b(K). \tag{2.4}$$

Let $Q = [0,1)^d$ be the unit cube, and let μ_0 be the Lebesgue measure on Q. Then the measure $\mathcal{T}^n\mu_0$ is the Lebesgue measure on the disjoint union

$$\mathcal{M}^nQ = \bigcup_{b_0,\dots,b_{n-1}\in B}\tau_{b_n}\dots\tau_{b_0}Q,$$

with renormalization factor $\frac{|\det R|^n}{N^n}$. Moreover, $\mathcal{T}^n\mu_0$ is a spectral measure with spectrum $\mathcal{S}^n\mathbb{Z}^d$.

Proof. First we check that the union that gives $\mathcal{M}^nQ$ is disjoint. If not, then there exist $b_0,\dots,b_{n-1}$ and $b_0',\dots,b_{n-1}'$ in B and $x,x' \in Q$ such that

$$R^{-n}(x+b_0+Rb_1+\dots+R^{n-1}b_{n-1}) = R^{-n}(x'+b_0'+Rb_1'+\dots+R^{n-1}b_{n-1}').$$

but this implies that $x - x' \in \mathbb{Z}^d$ and this is impossible, unless $x = x'$. Since (B,L) is a Hadamard pair, the points in B are incongruent $\mod R\mathbb{Z}^d$. And therefore we get $b_0 = b_0',\dots,b_{n-1} = b_{n-1}'$. Thus the union is disjoint.

Next we compute the measure $\mathcal{T}^n\mu$. It is enough to take $n = 1$, the general case is analogous. We have for f continuous on $\mathbb{R}^d$

$$\int f\,d\mathcal{T}\mu_0 = \frac{1}{N}\sum_{b\in B}\int_Q f(R^{-1}(x+b))\,dx = \frac{|\det R|}{N}\sum_{b\in B}\int_{\tau_b(Q)}f(y)\,dy.$$

This shows that $\mathcal{T}\mu_0$ is Lebesgue measure on $\mathcal{M}Q$ renormalized by $|\det R|/N$.

The fact that $\mathcal{S}^n\mathbb{Z}^d$ is a spectrum for $\mathcal{T}^n\mu_0$ follows from Proposition 2.2. $\square$

Proposition 2.7. *Let (B, L) be a Hadamard pair. Let Λ_0 be a subset of $\Pi(B)$ and assume that Λ_0 is the spectrum of some Borel probability measure μ_0 on $\mathbb{R}^d$ and $\mathcal{S}\Lambda_0 \subset \Lambda_0$. Then the set*

$$\Lambda := \bigcap_{n\geq 0} \mathcal{S}^n\Lambda_0$$

is orthogonal in $L^2(\mu_B)$.

Proof. For $n \in \mathbb{N}$, let $\mu_n := \mathcal{T}^n\mu_0$. With Proposition 2.2, we have that μ_n is a spectral measure with spectrum $\mathcal{S}^n\Lambda_0$.

Also, from [Hut81] we know that $\mu_n = \mathcal{T}^n\mu_0$ converges weakly to μ_B. Take two distinct λ_1, λ_2 in Λ. Then λ_1, λ_2 are in $\mathcal{S}^n\Lambda_0$ for all n, and since $\mathcal{S}^n\Lambda_0$ is a spectrum for μ_n, we have

$$\int e^{2\pi i(\lambda_1-\lambda_2)\cdot x}\, d\mu_n(x) = 0.$$

Since μ_n converges to μ_B weakly, we get that

$$\int e^{2\pi i(\lambda_1-\lambda_2)\cdot x}\, d\mu_B(x) = 0.$$

This proves that Λ is orthogonal in $L^2(\mu_B)$. $\square$

Theorem 2.8. *In dimension $d = 1$, suppose (B, L) is a Hadamard pair. Then the measure μ_B is a spectral measure with spectrum*

$$\Lambda = \bigcap_{n\geq 0} \mathcal{S}^n\left(\Pi(B)\right).$$

Proof. We use the result in [DJ06] that states that μ_B is a spectral measure with spectrum $\tilde{\Lambda}$ - the smallest $\mathcal{S}$-invariant set that contains $-C$ for all B-extreme L-cycles C.

Definition 2.9. Define the IFS

$$\tau_l^{(L)}(x) = (R^T)^{-1}(x + l), \quad (x \in \mathbb{R}^d).$$

To simplify the notation we will use just τ_l for $\tau_l^{(L)}$, the subscript l will indicate that we use the map $\tau_l^{(L)}$ and the subscript b will indicate that we use the map τ_b.

We say that a finite set $C = \{x_0, x_1, \dots, x_{p-1}\}$ is an L-cycle, if there exists $l_0, l_1, \dots, l_{p-1} \in L$ such that

$$\tau_l x_i = x_{i+1}, \quad i \in \{0, \dots, p-1\},$$

where $x_p := x_0$.

We say that this cycle is B-extreme if

$$|m_B(x_i)| = 1 \text{ for all } i \in \{0, \dots, p-1\}.$$

Thus, we have to prove only that $\Lambda = \cap_n \mathcal{S}^n\Pi(B)$ is equal to $\tilde{\Lambda}$.

We need the following lemma:

Lemma 2.10. [DJ07, Theorem 4.1] *Assume (B, L) is a Hadamard pair. Suppose there exist d linearly independent vectors in the set*

$$\Gamma(B) := \left\{ \sum_{k=0}^{n} R^k b_k : b_k \in B, n \in \mathbb{N} \right\}. \tag{2.5}$$

Define

$$\Gamma(B)^\circ := \left\{ x \in \mathbb{R}^d : \beta \cdot x \in \mathbb{Z} \text{ for all } \beta \in \Gamma(B) \right\}. \tag{2.6}$$

Then $\Gamma(B)^\circ$ is a lattice that contains $\mathbb{Z}^d$ which is invariant under R^T, and if $l, l' \in L$ with $l - l' \in R^T \Gamma(B)^\circ$ then $l = l'$. Moreover

$$\Gamma(B)^\circ \cap X_L \supset \bigcup \{ C : C \text{ is a } B\text{-extreme } L\text{-cycle} \}. \tag{2.7}$$

Note first that $\Pi(B) = \Gamma(B)^\circ$, because if $b\gamma \in \mathbb{Z}$ then $R^k b\gamma \in \mathbb{Z}$ for all $k \geq 0$.

Since L is contained in $\mathbb{Z}$, we have $\mathcal{S}\Gamma(B)^\circ \subset \Gamma(B)^\circ$ and this implies that $\mathcal{S}\Lambda \subset \Lambda$. Also, since all B-extreme L-cycles are contained in $\Gamma(B)^\circ$, and since $\mathcal{S}(-C) \supset -C$ if C is a cycle, we obtain that Λ contains all these cycles. Therefore $\tilde{\Lambda} \subset \Lambda$.

Now take $x_0 \in \Lambda$. Then, since $x_0 \in \mathcal{S}\Gamma(B)^\circ$, there is a $x_1 \in \Gamma(B)^\circ$ and $l_0 \in L$ such that $x_0 = Rx_1 + l_0$. Since elements in L are incongruent mod $R\Gamma(B)^\circ$, it follows that l_0 is uniquely determined by x_0. So x_1 is also uniquely determined; and, since $x_0 \in \Lambda$ it follows that $x_1 \in \Lambda$. This implies that $\tau_{l_0}(-x_0) = -x_1$. By induction we can find an infinite sequence $l_0, l_1, \ldots, l_n \ldots$ in L such that $x_{n+1} = -\tau_{l_n} \ldots \tau_{l_0}(-x_0)$ is in Λ. But then x_{n+1} converges to the attractor X_L of the IFS $(\tau_l)_{l \in L}$ and it is also a sequence in $\Lambda \subset \Gamma(B)^\circ$. Therefore from some point on the sequence has to be in a cycle inside $X_L \cap \Gamma(B)^\circ$. Since $|m_B(x)| = 1$ for $x \in \Gamma(B)^\circ$, we see that this implies that x_n will land in one of the B-extreme L-cycles.

Hence, every point x_0 can be obtained from a cycle point after the application of the operations $x \mapsto Rx + l$, $l \in L$. Therefore $\Lambda \subset \tilde{\Lambda}$.

□

Remark 2.11. Theorem 2.8 is false in higher dimensions. For example, take

$$R = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}, \quad B = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\} \quad L = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

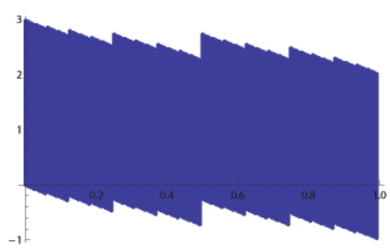

FIGURE 1. The attractor $X(B)$.

We proved in [DJ10] that the measure μ_B is the Lebesgue measure on the attractor $X(B)$ and it has spectrum $\mathbb{Z} \times \frac{1}{3}\mathbb{Z}$.

Note that the set

$$\Pi(B) = \{\gamma \in \mathbb{R}^2 : \gamma \cdot b \in \mathbb{Z} \text{ for all } b \in B\}$$

is in this case $\Pi(B) = \mathbb{Z} \times \frac{1}{3}\mathbb{Z}$.

It is easy to check that $(0, 2/3)^T$ is not in $\mathcal{S}\Pi(B)$ and therefore $\cap_{n\geq 0}\mathcal{S}^n\Pi(B)$ is a proper subset of $\Pi(B) = \mathbb{Z} \times \frac{1}{3}\mathbb{Z}$, thus it is incomplete.

3. Riesz bases, Bessel sequences and frames

In Proposition 3.2 and Theorem 3.4 we give characterization of Bessel spectra, first in terms of the Grammian matrix, for a general measure, and then in terms of some finite matrices for affine IFS measures. Theorem 3.4 is then reformulated in Theorem 3.8 in terms of an uniform bound for some finite atomic measures. In Theorem 3.9 we give a sufficient condition for a set to be a frame spectrum for an affine IFS with no overlap, in terms of the same finite matrices.

Proposition 3.1. *Suppose the sequence of Borel measures μ_n on $\mathbb{R}^d$ converges to the Borel measure μ weakly.*

(i) *Assume Λ_n is a Riesz basic sequence for μ_n with bounds A_n and B_n. Suppose finally that $\lim A_n = A > 0$, and $\lim B_n = B < \infty$. Then $\cap\Lambda_n$ is a Riesz basic sequence for μ (provided the intersection is nonempty).*

(ii) *Suppose Λ_n is a Schauder basic sequence in $L^2(\mu_n)$ with constant C_n. Suppose $\lim C_n = C < \infty$. Then $\cap\Lambda_n$ is a Schauder basic sequence in $L^2(\mu)$ with constant C.*

Proof. (i) Consider a finite set of K frequencies in $\cap\Lambda_n$, say $\lambda_1, \dots, \lambda_K$. For the measure μ_n, $\{e_{\lambda_1}, \dots, e_{\lambda_K}\} \subset L^2(\mu_n)$ is a Riesz basic sequence with bounds between A_n and B_n. Thus, the Grammian matrix M_n^K given by

$$M_n^K[i,j] = \langle e_{\lambda_i}, e_{\lambda_j}\rangle_{L^2(\mu_n)}$$

satisfies

$$A_n^2 I \leq M_n^K \leq B_n^2.$$

Note that the sequence M_n^K converges entry-wise to the Grammian matrix M^K given by

$$M^K[i,j] = \langle e_{\lambda_i}, e_{\lambda_j}\rangle_{L^2(\mu)}.$$

Given $\epsilon > 0$, find N such that the Frobenius norm $\|M^K - M_n^K\|_2 < \epsilon$ for $n > N$. Thus, $M^K - M_n^K$ is self-adjoint with norm less than ϵ, hence

$$-\epsilon I \leq M^K - M_n^K \leq \epsilon I.$$

Therefore,

$$A_n^2 I - \epsilon I \leq M_n^K + M^K - M_n^K \leq B_n^2 I + \epsilon I$$

and taking limits, we get

$$A^2 I - \epsilon I \leq M^K \leq B^2 I + \epsilon I.$$

This is true for any ϵ. Thus $\{e_{\lambda_1}, \dots, e_{\lambda_K}\} \subset L^2(\mu)$ is a Riesz basic sequence with bounds between A and B. Since $\{\lambda_1, \dots, \lambda_K\}$ were arbitrary, this is true for all of Λ.

(ii) Let $\|\cdot\|_n$ denote the norm in $L^2(\mu_n)$ and $\|\cdot\|$ denote the norm in $L^2(\mu)$. Note that any subset of Λ_n is also a Schauder basic sequence in $L^2(\mu_n)$ with basis constant no greater than C_n.

Fix constants $\{a_1, \dots, a_K\}$ and $1 \le k \le K$, and let $\{\lambda_1, \dots, \lambda_K\}$ be a finite subset of $\cap \Lambda_n$. We have

$$\left\| \sum_{j=1}^{k} a_j e_{\lambda_j} \right\|_n \le C_n \left\| \sum_{j=1}^{K} a_j e_{\lambda_j} \right\|_n$$

Taking limits, we get

$$\left\| \sum_{j=1}^{k} a_j e_{\lambda_j} \right\| \le C \left\| \sum_{j=1}^{K} a_j e_{\lambda_j} \right\|.$$

□

Next, we give some characterizations of Bessel spectra. One is in terms of the Gram matrix (Proposition 3.2) and it applies to general Borel measures. The other is in terms of the norm of some finite matrices (Theorem 3.4) and applies to affine IFS measures.

Proposition 3.2. *Let μ be a Borel probability measure on $\mathbb{R}^d$. Then a discrete subset Λ of $\mathbb{R}^d$ is a Bessel spectrum with bound M if and only if the matrix*

$$(\widehat{\mu}(\lambda - \lambda'))_{\lambda, \lambda' \in \Lambda}$$

defines a bounded operator on $l^2(\Lambda)$ with norm less than M.

Proof. By [Chr03, Lemma 3.5.1], $(e_\lambda)_{\lambda \in \Lambda}$ is a Bessel sequence with bound M if and only if its Gram matrix

$$(\langle e_\lambda \, , \, e_{\lambda'} \rangle)_{\lambda, \lambda' \in \Lambda}$$

defines a bounded operator on $l^2(\Lambda)$ with norm less than M. But $\langle e_\lambda \, , \, e'_\lambda \rangle = \widehat{\mu}(\lambda - \lambda')$ for all $\lambda, \lambda' \in \Lambda$. □

Then, an application of Schur's lemma gives the following (see [Chr03, Proposition 3.5.4]):

Proposition 3.3. *Let μ be a Borel probability measure on $\mathbb{R}^d$. Let Λ be discrete subset of $\mathbb{R}^d$. If there exists a constant $M > 0$ such that*

$$\sum_{\lambda' \in \Lambda} |\widehat{\mu}(\lambda - \lambda')| \le M \text{ for all } \lambda \in \Lambda,$$

then Λ is a Bessel spectrum with Bessel bound M.

Notation. For $k = (k_1, \dots, k_n) \in B^n$ we define

$$\tau_k := \tau_{k_n} \circ \cdots \circ \tau_{k_1}.$$

Theorem 3.4. *Assume the measure μ associated to the affine IFS $(\tau_b)_{b \in B}$ has no overlap. Let Λ be a discrete subset of $\mathbb{R}^d$. Then the following assertions are equivalent:*

(i) *The set Λ is a Bessel spectrum for μ.*
(ii) *There exist $r_0 > 0$ and $C > 0$ with the following property: if for all $n \in \mathbb{N}$ we define*

$$\Lambda_n(r_0) := \{\lambda \in \Lambda : |R^{T^{-n}} \lambda| \le r_0\}$$

then the norm of the matrix

$$\frac{1}{\sqrt{N^n}} \left(e^{-2\pi i \lambda \cdot \tau_k(0)} \right)_{\lambda \in \Lambda_n(r_0), k \in B^n} \tag{3.1}$$

is bounded by C.

(iii) *For all $r_0 > 0$ there exists a $C(r_0) > 0$ such that for all $n \in \mathbb{N}$ the norm of the matrix in (3.1) is bounded by $C(r_0)$.*

We begin with some lemmas.

Lemma 3.5. [DHSW11, Lemma2.2] *Let $X = X_B$ be the attractor of the IFS $(\tau_b)_{b\in B}$. For all $n \in \mathbb{N}$ and $k \in B^n$*

$$\int_{\tau_k(X)} f\,d\mu = \frac{1}{N^n}\int_X f\circ\tau_k\,d\mu, \quad (f \in L^\infty(X)).$$

Lemma 3.6. *Let $f = \sum_{k\in B^n} c_k\chi_{\tau_k(X)}$. Then*

$$\langle f\,,\,e_\lambda\rangle = \frac{1}{N^n}\widehat{\mu}(-R^{T-n}\lambda)\sum_{k\in B^n} c_k e^{-2\pi i\lambda\cdot\tau_k(0)}, \tag{3.2}$$

$$\|f\|^2 = \frac{1}{N^n}\sum_{k\in B^n}|c_k|^2. \tag{3.3}$$

Proof. We have, using Lemma 3.5:

$$E := \langle f\,,\,e_\lambda\rangle = \sum_{k\in B^n} c_k\int_{\tau_k(X)} e^{-2\pi i\lambda\cdot x}\,d\mu(x) = \sum_{k\in B^n} c_k\frac{1}{N^n}\int_X e^{-2\pi i\lambda\cdot\tau_k(x)}\,d\mu(x)$$

and since $\tau_k(x) = \tau_k(0) + R^{-n}x$:

$$E = \frac{1}{N^n}\sum_{k\in B^n} c_k e^{-2\pi i\lambda\cdot\tau_k(0)}\int_X e^{-2\pi i\lambda\cdot R^{-n}x}\,d\mu(x)$$

and (3.2) follows.

Equation (3.3) can be obtained from a simple computation (since μ has no overlap, the measure of $\tau_k(X)$ is $1/N^n$). ☐

Proof of Theorem 3.4. (i) ⇒ (ii). Since $\widehat{\mu}$ is continuous and $\widehat{\mu}(0) = 1$, there exist $\delta > 0$ and $r_0 > 0$ such that $|\widehat{\mu}(x)|^2 \geq \delta$ if $|x| \leq r_0$.

Using Lemma 3.6 and the Bessel inequality, we have for any n and any f of the form $f = \sum_{k\in\mathbb{N}} c_k\chi_{\tau_k(X)}$:

$$M\frac{1}{N^n}\sum_{k\in B^n}|c_k|^2 = M\|f\|^2 \geq \sum_{\lambda\in\Lambda_n(r_0)}|\langle f\,,\,e_\lambda\rangle|^2 = \sum_{\lambda\in\Lambda_n(r_0)}\frac{1}{N^{2n}}|\widehat{\mu}(-R^{T-n}\lambda)|^2\left|\sum_{k\in B^n} c_k e^{-2\pi i\lambda\cdot\tau_k(0)}\right|^2.$$

But, if $\lambda \in \Lambda_n(r_0)$, then $|R^{T-n}\lambda| \leq r_0$ so $|\widehat{\mu}(R^{T-n}\lambda)|^2 \geq \delta$. Therefore

$$M\sum_{k\in B^n}|c_k|^2 \geq \delta\sum_{\lambda\in\Lambda_n(r_0)}\left|\frac{1}{\sqrt{N^n}}\sum_{k\in B^n} c_k e^{-2\pi i\lambda\cdot\tau_k(0)}\right|^2$$

which shows that the norm of the matrix in (3.1) is less than $\sqrt{M/\delta}$.

(ii) ⇒ (i) It is enough to prove the Bessel bound for functions of the form $f = \sum_{k\in B^m} c_k\chi_{\tau_k(X)}$ because these are dense in $L^2(\mu)$.

To see that these functions are dense in $L^2(\mu)$, take first a continuous function f on $X = X_B$ and $\epsilon > 0$. Since X is compact, the function f is uniformly continuous. Take m large enough

such that the diameter of all sets $\tau_k(X)$, $k\in B^m$, is small enough so that $|f(x)-f(y)|<\epsilon$ for all $x,y\in\tau_k(X)$ and all $k\in B^m$. Then, using the non-overlap, define

$$g:=\sum_{k\in B^m} f(\tau_k(0))\chi_{\tau_k(X)}.$$

It is easy to see that $\sup_{x\in X}|f(x)-g(x)|\leq\epsilon$. This proves that these step functions are dense in $C(X)$, and since μ is a regular Borel measure, they are dense in $L^2(\mu)$.

If f is of this form and $n\geq m$ then we can say f is also of the form $f=\sum_{k\in B^n}c_k\chi_{\tau_k(X)}$ because $(\tau_k(X))_{k\in B^n}$ is a refinement of the family of sets $(\tau_k(X))_{k\in B^m}$.

So take $n\geq m$. We have, using Lemma 3.6,

$$\sum_{\lambda\in\Lambda_n(r_0)}|\langle f\,,\,e_\lambda\rangle|^2=\frac{1}{N^{2n}}\sum_{\lambda\in\Lambda_n(r_0)}|\widehat{\mu}(-R^{T^{-n}}\lambda)|^2\left|\sum_{k\in B^n}c_ke^{-2\pi i\lambda\cdot\tau_k(0)}\right|^2$$

and, since $|\widehat{\mu}|\leq 1$, and using the hypothesis, we get further

$$\leq\frac{1}{N^{2n}}\sum_{\lambda\in\Lambda_n(r_0)}\left|\sum_{k\in B^n}c_ke^{-2\pi i\lambda\cdot\tau_k(0)}\right|^2\leq C\frac{1}{N^n}\sum_{k\in B^n}|c_k|^2=C\|f\|^2.$$

Since n was arbitrary, we obtain

$$\sum_{\lambda\in\Lambda}|\langle f\,,\,e_\lambda\rangle|^2\leq C\|f\|^2.$$

This proves (i).

$(iii)\Rightarrow(ii)$ is clear.

$(ii)\Rightarrow(iii)$. Take $r_1>0$. Since $R^{T^{-1}}$ is contractive, there exists $m\in\mathbb{N}$ with the property that $R^{T^{-m}}B(0,r_1)\subset B(0,r_0)$. Then, for all n we have

$$\Lambda_n(r_1):=\{\lambda\in\Lambda:|R^{T^{-n}}\lambda|\leq r_1\}\subset\{\lambda\in\Lambda:|R^{T^{-(n+m)}}\lambda|\leq r_0\}.$$

Also, since $0\in B$, we have

$$\{\tau_k(0):k\in B^n\}\subset\{\tau_k(0):k\in B^{n+m}\}.$$

Then the matrix

$$A(n,r_1):=\left(e^{-2\pi i\lambda\cdot\tau_k(0)}\right)_{\lambda\in\Lambda_n(r_1),k\in B^n}$$

is a submatrix of the matrix

$$A(n+m,r_0):=\left(e^{-2\pi i\lambda\cdot\tau_k(0)}\right)_{\lambda\in\Lambda_{n+m}(r_0),k\in B^{n+m}}.$$

Therefore, for all n

$$\left\|\frac{1}{\sqrt{N^n}}A(n,r_1)\right\|=\sqrt{N^m}\left\|\frac{1}{\sqrt{N^{n+m}}}A(n,r_1)\right\|\leq\sqrt{N^m}\left\|\frac{1}{\sqrt{N^{n+m}}}A(n+m,r_0)\right\|\leq\sqrt{N^m}C(r_0).$$

This proves (iii). ∎

Conditions (ii) and (iii) in Theorem 3.4 can be expressed in terms of Bessel sequences for finite atomic measures. We formulate this in the following lemma:

Lemma 3.7. *Let F be a finite subset of $\mathbb{R}^d$ and let $\delta_F := \frac{1}{\#F}\sum_{f\in F}\delta_f$, where δ_f is the Dirac measure at f. Let G be a finite subset of $\mathbb{R}^d$. Then G is a Bessel spectrum for δ_F with bound M if and only if the matrix*

$$\frac{1}{\sqrt{\#F}}\left(e^{2\pi i f\cdot g}\right)_{f\in F, g\in G}$$

has norm less than $\sqrt{M}$.

Proof. Let A be this matrix. Then, for all $g, g' \in G$:

$$(AA^*)_{g,g'} = \frac{1}{\#F}\sum_{f\in F} e^{2\pi i f\cdot(g-g')} = \widehat{\delta}_F(g-g').$$

So, using Proposition 3.2 and the fact that $\|A^*A\| = \|A\|^2$, the result follows. □

Therefore, Theorem 3.4 can be reformulated as follows:

Theorem 3.8. *Using the notations from Theorem 3.4, the following assertions are equivalent:*

(i) *The set Λ is a Bessel spectrum for μ.*
(ii) *There exist $r_0 > 0$ and $C > 0$ with the property that for all n, the set $\Lambda_n(r_0)$ is a Bessel sequence with bound C for the atomic measure $\delta_{B^n} := \frac{1}{N^n}\sum_{k\in B^n}\delta_{\tau_k(0)}$.*
(iii) *For all $r_0 > 0$ there exists a constant $C(r_0) > 0$ with the property that for all n, the set $\Lambda_n(r_0)$ is a Bessel sequence with bound $C(r_0)$ for the atomic measure $\delta_{B^n} := \frac{1}{N^n}\sum_{k\in B^n}\delta_{\tau_k(0)}$.*

In the next theorem we will give a sufficient condition for a set to be a frame spectra formulated again in terms of the finite matrices in (3.1).

Theorem 3.9. *Assume that the measure μ associated to the affine IFS $(\tau_b)_{b\in B}$ has no overlap. Let Λ be a discrete subset of $\mathbb{R}^d$. Let $r_0 > 0$ and $\delta > 0$ be such that $|\widehat{\mu}(x)| \geq \delta$ for $|x| \leq r_0$. Define*

$$\Lambda_n(r_0) := \{\lambda \in \Lambda : |R^{T^{-n}}\lambda| \leq r_0\}$$

and define the matrix A_n

$$A_n := \frac{1}{\sqrt{N^n}}\left(e^{-2\pi i\lambda\cdot\tau_k(0)}\right)_{\lambda\in\Lambda_n(r_0), k\in B^n}.$$

Suppose there exist constants $m, M > 0$ such that

$$m\|f\|^2 \leq \|A_n f\|^2 \leq M\|f\|^2$$

for all $f \in \mathbb{C}^{\#B^n}$ and all $n \in \mathbb{N}$. Then Λ is a frame spectrum for μ.

Proof. The upper frame bound follows from Theorem 3.4. For the lower frame bound we use the computations in the proof of Theorem 3.4(i)⇒(ii).

We have for any n and any f of the form $f = \sum_{k\in B^n} c_k\chi_{\tau_k(X)}$:

$$\sum_{\lambda\in\Lambda_n(r_0)} |\langle f\,,\, e_\lambda\rangle|^2 \geq \frac{1}{N^n}\delta\sum_{\lambda\in\Lambda_n(r_0)}\left|\frac{1}{\sqrt{N^n}}\sum_{k\in B^n} c_k e^{-2\pi i\lambda\cdot\tau_k(0)}\right|^2.$$

Using the hypothesis this is bigger than

$$\frac{\delta m}{N^n}\sum_{k\in B^n}|c_k|^2 = m\delta\|f\|^2.$$

The density of the functions f of this type proves that Λ is a frame. ☐

Example 3.10. Let $\tau_0(x) = \dfrac{x}{3}$ and $\tau_1(x) = \dfrac{x+2}{3}$. We consider the following sequence of closed sets:

$$\Omega_0 = [0,1]; \qquad \Omega_{n+1} = \tau_0(\Omega_n)\cup\tau_1(\Omega_n).$$

Note that $\cap_n\Omega_n = C_3$, the middle third Cantor set. We also endow these sets with a measure ν_n where ν_n is Lebesgue measure restricted to Ω_n, normalized (by $(3/2)^n$) so that $\|\nu_n\| = 1$. As in the proof of Corollary 2.6, we have $\mathcal{T}^n\nu_0 = \nu_n$. Therefore, this sequence of measures converges weakly to the measure μ_3. (See e.g. [Hut81])

We consider the IFS

$$\rho_0(x) = 3x; \qquad \rho_1(x) = 3x+1 \tag{3.4}$$

acting on $\mathbb{Z}$. We also consider the sequence of spectra given as:

$$\Gamma_0 = \mathbb{Z}; \qquad \Gamma_{n+1} = \rho_0(\Gamma_n)\cup\rho_1(\Gamma_n).$$

Proposition 3.11. *The sequence Γ_n is a Riesz basic spectrum for the measure ν_n, with basis bounds A_n, B_n which satisfy*

$$A^n \le A_n \le B_n \le B^n,$$

where $A = \sqrt{\frac{1}{2}}$ and $B = \sqrt{\frac{3}{2}}$ are the basis bounds for the columns of the matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & e^{4\pi i/3}\end{pmatrix}.$$

Moreover, the Riesz basic sequence Γ_n is complete in $L^2(\nu_n)$.

Proof. The basis bounds A and B can be computed by taking the roots of the largest and smallest eigenvalues of U^*U where U is the given matrix.

Suppose Γ_{n-1} is a Riesz basic spectrum for ν_{n-1} with bounds C and D. Let $\{a_k\}_{k\in\Gamma_n}$ be a compactly supported sequence.

$$\begin{aligned}\int_{\Omega_n}|\sum_{k\in\Gamma_n}a_ke_k(x)|^2d\nu_n &= \int_{\tau_0(\Omega_{n-1})}|\sum_{k\in\Gamma_n}a_ke_k(x)|^2d\nu_n + \int_{\tau_1(\Omega_{n-1})}|\sum_{k\in\Gamma_n}a_ke_k(x)|^2d\nu_n\\ &= \int_{\tau_0(\Omega_{n-1})}|\sum_{k\in\rho_0(\Gamma_{n-1})}a_ke_k(x) + \sum_{k\in\rho_1(\Gamma_{n-1})}a_ke_k(x)|^2d\nu_n\\ &\quad + \int_{\tau_1(\Omega_{n-1})}|\sum_{k\in\rho_0(\Gamma_{n-1})}a_ke_k(x) + \sum_{k\in\rho_1(\Gamma_{n-1})}a_ke_k(x)|^2d\nu_n.\end{aligned}$$

Let $f(x) = \sum_{k\in\rho_0(\Gamma_{n-1})}a_ke_k(x)$ and $g(x) = \sum_{k\in\rho_1(\Gamma_{n-1})}a_ke_{k-1}(x)$ so that both $f(x)$ and $g(x)$ are finite linear combinations of $\{e_k : k\in\rho_0(\Gamma_{n-1})\}$. Since $\Gamma_{n-1}\subset\mathbb{Z}$, we have $\rho_0(\Gamma_{n-1})\subset 3\mathbb{Z}$, so

$f(x+2/3) = f(x)$ and $g(x+2/3) = g(x)$. Continuing with the above computation:

$$\begin{aligned}\int_{\Omega_n} \Big|\sum_{k\in\Gamma_n} a_k e_k(x)\Big|^2 d\nu_n &= \int_{\tau_0(\Omega_{n-1})} |f(x)+e_1(x)g(x)|^2 d\nu_n + \int_{\tau_1(\Omega_{n-1})} |f(x)+e_1(x)g(x)|^2 d\nu_n \\ &= \int_{\tau_0(\Omega_{n-1})} |f(x)+e_1(x)g(x)|^2 + |f(x+\frac{2}{3}) + e_1(x+\frac{2}{3})g(x+\frac{2}{3})|^2 d\nu_n \\ &= \int_{\tau_0(\Omega_{n-1})} |f(x)+e_1(x)g(x)|^2 + |f(x) + e_1(x+\frac{2}{3})g(x)|^2 d\nu_n \\ &= \int_{\tau_0(\Omega_{n-1})} \left\| \begin{pmatrix} 1 & 1 \\ 1 & e_1(\frac{2}{3}) \end{pmatrix} \begin{pmatrix} f(x) \\ e_1(x)g(x) \end{pmatrix} \right\|^2 d\nu_n.\end{aligned}$$

By hypothesis, we have that

$$2A^2(|f(x)|^2 + |e_1(x)g(x)|^2) \leq \left\| \begin{pmatrix} 1 & 1 \\ 1 & e_1(\frac{2}{3}) \end{pmatrix} \begin{pmatrix} f(x) \\ e_1(x)g(x) \end{pmatrix} \right\|^2 \leq 2B^2(|f(x)|^2 + |e_1(x)g(x)|^2).$$

Therefore,

$$2A^2\Big(\int_{\tau_0(\Omega_{n-1})} |f(x)|^2 + |g(x)|^2 d\nu_n\Big) \leq \int_{\Omega_n} \Big|\sum_{k\in\Gamma_n} a_k e_k(x)\Big|^2 d\nu_n \leq 2B^2\Big(\int_{\tau_0(\Omega_{n-1})} |f(x)|^2 + |g(x)|^2 d\nu_n\Big). \tag{3.5}$$

For $f(x) = \sum_{k\in\rho_0(\Gamma_{n-1})} a_k e_k(x) = \sum_{k\in\Gamma_{n-1}} a_{\rho_0(k)} e_k(x)$, we have

$$\begin{aligned}\int_{\tau_0(\Omega_{n-1})} \Big|\sum_{k\in\rho_0(\Gamma_{n-1})} a_k e_k(x)\Big|^2 d\nu_n &= \int_{\tau_0(\Omega_{n-1})} \Big|\sum_{k\in\Gamma_{n-1}} a_{\rho_0(k)} e_k(\rho_0(x))\Big|^2 d\nu_n \\ &= \frac{1}{2}\int_{\Omega_{n-1}} \Big|\sum_{k\in\Gamma_{n-1}} a_{\rho_0(k)} e_k(x)\Big|^2 d\nu_{n-1}.\end{aligned} \tag{3.6}$$

Since Γ_{n-1} is a Riesz basic spectrum for ν_{n-1}, we have

$$A_{n-1}^2 \left(\sum_{k\in\Gamma_{n-1}} |a_{\rho_0(k)}|^2\right) \leq \int_{\Omega_{n-1}} |f|^2 d\nu_{n-1} \leq B_{n-1}^2 \left(\sum_{k\in\Gamma_{n-1}} |a_{\rho_0(k)}|^2\right) \tag{3.7}$$

Likewise, we have

$$A_{n-1}^2 \left(\sum_{k\in\Gamma_{n-1}} |a_{\rho_1(k)}|^2\right) \leq \int_{\Omega_{n-1}} |g|^2 d\nu_{n-1} \leq B_{n-1}^2 \left(\sum_{k\in\Gamma_{n-1}} |a_{\rho_1(k)}|^2\right)$$

Combining (3.5), (3.6), and (3.7), we obtain

$$\begin{aligned}
A_{n-1}^2 A^2 (\sum_{k\in\Gamma_n} |a_k|^2) &= A_{n-1}^2 A^2 (\sum_{k\in\Gamma_{n-1}} |a_{\rho_0(k)}|^2 + \sum_{k\in\Gamma_{n-1}} |a_{\rho_1(k)}|^2) \\
&\le A^2 (\int_{\Omega_{n-1}} |f|^2 + |g|^2 d\nu_{n-1}) \\
&\le \int_{\Omega_n} |\sum_{k\in\Gamma_n} a_k e_k(x)|^2 d\nu_n \\
&\le B^2 (\int_{\Omega_{n-1}} |f|^2 + |g|^2 d\nu_{n-1}) \\
&\le B_{n-1} B^2 (\sum_{k\in\Gamma_n} |a_k|^2).
\end{aligned}$$

This completes the proof that Γ_n is a Riesz basic spectrum for ν_n.

We now need to prove that Γ_n is complete in $L^2(\nu_n)$. Again we proceed by induction, and we will show that Γ_n has uniformly dense span in $C(\Omega_n)$. Let $g \in C(\Omega_n)$, and define the functions

$$k_1(x) = \alpha g(x) + \beta g(x + 2/3); \qquad k_2(x) = e^{-2\pi i x}(\gamma g(x) + \delta g(x + 2/3)),$$

where

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & e^{4\pi i/3} \end{pmatrix}^{-1} =: A^{-1}.$$

Note that $k_1, k_2 \in C(\Omega_n)$, so for $x \in \Omega_{n-1}$ the functions defined as $h_1(x) := k_1(x/3)$ and $h_2(x) := k_2(x/3)$ are both elements of $C(\Omega_{n-1})$. By hypothesis, there exists $l_1, l_2 \in span(\Gamma_{n-1})$ such that for all $x \in \Omega_{n-1}$, $|h_1(x) - l_1(x)| < \epsilon$ and $|h_2(x) - l_2(x)| < \epsilon$. Therefore $l_1(3x) \in span(\rho_0(\Gamma_{n-1}))$ and $e^{2\pi i x} l_2(3x) \in span(\rho_1(\Gamma_{n-1}))$. Note that for $q(x) \in span(\rho_0(\Gamma_{n-1}))$, $q(x+2/3) = q(x)$, and for $q(x) \in span(\rho_1(\Gamma_{n-1}))$, $q(x + 2/3) = e^{4\pi i/3} q(x)$.

We consider $f(x) = l_1(3x) + e^{2\pi i x} l_2(3x) \in span(\Gamma_n)$. For any $x \in \tau_0(\Omega_{n-1})$,

$$\begin{aligned}
|g(x) - f(x)| + |g(x + 2/3) - f(x + 2/3)| &= \left\| \begin{pmatrix} g(x) \\ g(x+2/3) \end{pmatrix} - \begin{pmatrix} l_1(3x) + e^{2\pi i x} l_2(3x) \\ l_1(3x) + e^{4\pi i/3} e^{2\pi i x} l_2(3x) \end{pmatrix} \right\|_1 \\
&= \left\| \begin{pmatrix} g(x) \\ g(x+2/3) \end{pmatrix} - A \begin{pmatrix} l_1(3x) \\ e^{2\pi i x} l_2(3x) \end{pmatrix} \right\|_1 \\
&= \left\| A \left(A^{-1} \begin{pmatrix} g(x) \\ g(x+2/3) \end{pmatrix} - \begin{pmatrix} l_1(3x) \\ e^{2\pi i x} l_2(3x) \end{pmatrix} \right) \right\|_1 \\
&\le \|A\|_1 \left\| \left(\begin{pmatrix} k_1(x) \\ e^{2\pi i x} k_2(x) \end{pmatrix} - \begin{pmatrix} l_1(3x) \\ e^{2\pi i x} l_2(3x) \end{pmatrix} \right) \right\|_1 \\
&= \|A\|_1 \left\| \left(\begin{pmatrix} h_1(3x) \\ e^{2\pi i x} h_2(3x) \end{pmatrix} - \begin{pmatrix} l_1(3x) \\ e^{2\pi i x} l_2(3x) \end{pmatrix} \right) \right\|_1 \\
&= \|A\|_1 \left(|h_1(3x) - l_1(3x)| + |h_2(3x) - l_2(3x)| \right) \\
&< 2\|A\|_1 \epsilon.
\end{aligned}$$

Therefore, g is in the uniform closure of $span(\Gamma_n)$. $\square$

Remark 3.12. Note that $\cap_n \Gamma_n = \{0, 1, 3, 4, \dots\} = \{\sum_{k=1}^{K} l_k 3^k : l_k = 0, 1\}$. By [DHSW11], this system cannot be a Bessel spectrum for μ_3, and so B_n diverges to ∞. A natural question follows: is $\cap_n \Gamma_n$ a Schauder basic spectrum for μ_3?

Remark 3.13. Note that Γ_n are determined by the "dual" iterated function system given by ρ_0 and ρ_1 in Equation 3.4. Indeed,

$$\Gamma_n = \cup_{k=1}^{2^n} 3^n \mathbb{Z} + q_k$$

where $q_k \in \{0, 1, \dots, 3^n - 1\}$, and correspond to the orbit of 0 under words of ρ_0, ρ_1 of length n.

Consider now the possibility of freely choosing the q_k's at each scale n: for each n, choose $Q_n \subset \{0, 1, \dots, 3^n - 1\}$ of size r_n (not necessarily 2^n) and let

$$\Gamma_n = \cup_{k=1}^{r_n} 3^n \mathbb{Z} + q_k.$$

Let $P_n = \{p_1 < p_2 < \cdots < p_{2^n}\}$ be the integers that appear as left endpoints for the intervals of the set $3^n \Omega_n$. The set P_n can be defined recursively as a set of integers $P_n \subset \{0, 1, \dots, 3^n - 1\}$ by $P_n = \cup_{p \in P_{n-1}} \{p, p + 2 \cdot 3^n\}$, $P_1 = \{0, 2\}$.

Theorem 3.14. *Suppose Q_n is such that the $2^n \times r_n$ matrix*

$$M(l, m) = e^{2\pi i q_l p_m / 3^n}$$

is bounded in norm by $2^{\frac{n}{2}} L$, for some L independent of n. (For convenience, suppose $p_k < p_{k+1}$ and $q_k < q_{k+1}$.) Then

$$\Gamma_n = \cup_{k=1}^{r_n} 3^n \mathbb{Z} + q_k$$

is a Bessel spectrum for ν_n with Bessel bound no greater than L^2 and hence $\cap \Gamma_n$ is a Bessel spectrum for μ_3 with Bessel bound no greater than L^2.

If in addition, the matrix M is bounded below by $2^{\frac{n}{2}} L'$, for some L' independent of n, then Γ_n is a Riesz basic spectrum for ν_n with lower basis bound L', and hence $\cap \Gamma_n$ is a Riesz basic spectrum for μ_3.

Proof. Fix n and let $\{c_\gamma\}_{\gamma \in \Gamma_n}$ be a square summable sequence indexed by Γ_n. We need to estimate the norm $\| \sum_{\gamma \in \Gamma_n} c_\gamma e_\gamma \|_{\nu_n}^2$. Decompose the sum as follows:

$$\sum_{\gamma \in \Gamma_n} c_\gamma e_\gamma(x) = \sum_{l=1}^{r_n} \sum_{z \in \mathbb{Z}} c_{3^n z + q_l} e_{3^n z + q_l}(x) = \sum_{l=1}^{r_n} \sum_{z \in \mathbb{Z}} d_{3^n z}^{l} e_{3^n z}(x) e_{q_l}(x) = \sum_{l=1}^{r_n} f_l(x) e_{q_l}(x)$$

where $d^l_{3^n z} = c_{3^n z + q_l}$ and $f_l(x) = \sum_{z\in\mathbb{Z}} d^l_{3^n z} e_{3^n z}(x)$. Note that $f_l(x + \frac{1}{3^n}) = f_l(x)$. We have that

$$\begin{aligned}
\| \sum_{\gamma\in\Gamma_n} c_\gamma e_\gamma \|^2_{\nu_n} &= \int | \sum_{\gamma\in\Gamma_n} c_\gamma e_\gamma(t)|^2 d\nu_n(t) \\
&= \frac{3^n}{2^n} \sum_{m=1}^{2^n} \int_0^{\frac{1}{3^n}} | \sum_{\gamma\in\Gamma_n} c_\gamma e_\gamma (t + \frac{p_m}{3^n})|^2 d(t) \\
&= \frac{3^n}{2^n} \sum_{m=1}^{2^n} \int_0^{\frac{1}{3^n}} | \sum_{l=1}^{r_n} f_l(x + \frac{p_m}{3^n}) e_{q_l}(x + \frac{p_m}{3^n})|^2 d(t) \\
&= \frac{3^n}{2^n} \sum_{m=1}^{2^n} \int_0^{\frac{1}{3^n}} | \sum_{l=1}^{r_n} f_l(x) e_{q_l}(x + \frac{p_m}{3^n})|^2 d(t) \\
&= \frac{3^n}{2^n} \int_0^{\frac{1}{3^n}} \left\| \begin{pmatrix} \sum_{l=1}^{r_n} f_l(x) e_{q_l}(x + \frac{p_1}{3^n}) \\ \vdots \\ \sum_{l=1}^{r_n} f_l(x) e_{q_l}(x + \frac{p_{2^n}}{3^n}) \end{pmatrix} \right\|^2 d(t) \\
&= \frac{3^n}{2^n} \int_0^{\frac{1}{3^n}} \left\| \begin{pmatrix} e_{q_1}(\frac{p_1}{3^n}) & \dots & e_{q_{r_n}}(\frac{p_1}{3^n}) \\ \vdots & \ddots & \vdots \\ e_{q_1}(\frac{p_{2^n}}{3^n}) & \dots & e_{q_{r_n}}(\frac{p_{2^n}}{3^n}) \end{pmatrix} \begin{pmatrix} f_1(x) e_{q_1}(x) \\ \vdots \\ f_{r_n}(x) e_{q_{r_n}}(x) \end{pmatrix} \right\|^2 d(t) \\
&\le \frac{3^n}{2^n} \int_0^{\frac{1}{3^n}} 2^n L^2 \left\| \begin{pmatrix} f_1(x) e_{q_1}(x) \\ \vdots \\ f_{r_n}(x) e_{q_{r_n}}(x) \end{pmatrix} \right\|^2 d(t) \\
&= 3^n L^2 \int_0^{\frac{1}{3^n}} \left\| \begin{pmatrix} f_1(x) \\ \vdots \\ f_{r_n}(x) \end{pmatrix} \right\|^2 d(t) \\
&= 3^n L^2 \left(3^{-n} \sum_{l=1}^{r_n} \sum_{z\in\mathbb{Z}} |d^l_{3^n z}|^2 \right) \\
&= L^2 \left(\sum_{\gamma\in\Gamma_n} |c_\gamma|^2 \right).
\end{aligned}$$

If the matrix is bounded below, the inequality above is reversed, with L replaced by L'.

□

References

[Chr03] Ole Christensen. *An introduction to frames and Riesz bases*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston Inc., Boston, MA, 2003.

[DHSW11] Dorin Ervin Dutkay, Deguang Han, Qiyu Sun, and Eric Weber. On the Beurling dimension of exponential frames. *Advances in Math.*, 226(1):285–297, 2011.

[DJ06] Dorin Ervin Dutkay and Palle E. T. Jorgensen. Iterated function systems, Ruelle operators, and invariant projective measures. *Math. Comp.*, 75(256):1931–1970 (electronic), 2006.

[DJ07] Dorin Ervin Dutkay and Palle E. T. Jorgensen. Analysis of orthogonality and of orbits in affine iterated function systems. *Math. Z.*, 256(4):801–823, 2007.

[DJ10] Dorin Ervin Dutkay and Palle E.T. Jorgensen. Spectral measures and Cuntz algebras. *preprint*, 2010.

[DS52] R. Duffin and A. Schaeffer. A class of nonharmonic Fourier series. *Trans. Amer. Math. Soc.*, 72:341–366, 1952.

[Hut81] John E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.*, 30(5):713–747, 1981.

[Jaf91] S. Jaffard. A density criterion for frames of complex exponentials. *Michigan Math. J.*, 38(3):339–348, 1991.

[JP98] Palle E. T. Jorgensen and Steen Pedersen. Dense analytic subspaces in fractal L^2-spaces. *J. Anal. Math.*, 75:185–228, 1998.

[OCS02] Joaquim Ortega-Cerdà and Kristian Seip. Fourier frames. *Ann. of Math. (2)*, 155(3):789–806, 2002.

[PW87] R. Paley and N. Wiener. *Fourier transforms in the complex domain*, volume 19 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1987. Reprint of the 1934 original.

[Sei95] Kristian Seip. On the connection between exponential bases and certain related sequences in $L^2(-\pi, \pi)$. *J. Funct. Anal.*, 130(1):131–160, 1995.

[Sin70] Ivan Singer. *Bases in Banach spaces. I.* Springer-Verlag, New York, 1970. Die Grundlehren der mathematischen Wissenschaften, Band 154.

[Dorin Ervin Dutkay] University of Central Florida, Department of Mathematics, 4000 Central Florida Blvd., P.O. Box 161364, Orlando, FL 32816-1364, U.S.A.,

E-mail address: ddutkay@mail.ucf.edu

[Deguang Han]University of Central Florida, Department of Mathematics, 4000 Central Florida Blvd., P.O. Box 161364, Orlando, FL 32816-1364, U.S.A.,

E-mail address: deguang.han@.ucf.edu

[Eric Weber]Department of Mathematics, 396 Carver Hall, Iowa State University, Ames, IA 50011, U.S.A.,

E-mail address: esweber@iastate.edu